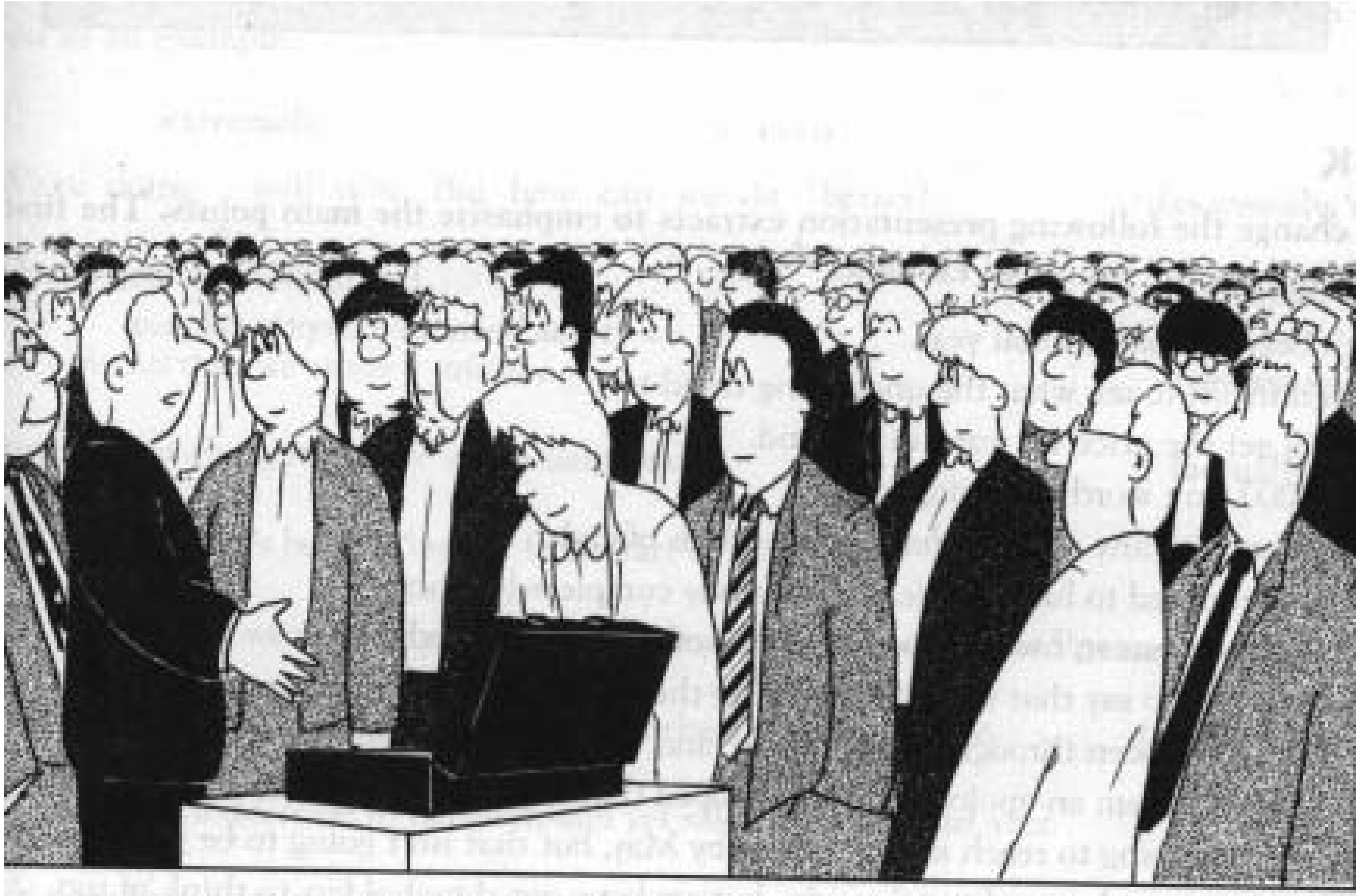
Choix des a priori de variance/ Illustration en Winbugs/Openbugs



"It's nice to see so many of you, but I've only got 20 samples."

Exemple/Données Pothoff

Growth measurements in 11 girls and 16 boys: Pothoff and Roy,1964; Little and Rubin, 1987

	Age (years)					Age (years)			
Girl	8	10	12	14	Boy	8	10	12	14
1	210	200	215	230	1	260	250	290	310
2	210	215	240	255	2	215		230	265
3	205		245	260	3	230	225	240	275
4	235	245	250	265	4	255	275	265	270
5	215	230	225	235	5	200		225	260
6	200		210	225	6	245	255	270	285
7	215	225	230	250	7	220	220	245	265
8	230	230	235	240	8	240	215	245	255
9	200		220	215	9	230	205	310	260
10	165		190	195	10	275	280	310	315
11	245	250	280	280	11	230	230	235	250
					12	215		240	280
					13	170		260	295
					14	225	255	255	260
					15	230	245	260	300
					16	220		235	250

distance from the centre of the pituary to the pteryomaxillary fissure (unit 10^{-4}m)

Modèle/Ecriture

i: indice de l'individu $i = 1,..,I = 25$ (11F+16G-2outliers)

j: indice de la mesure à l'age t_j

modèle hiérarchique

$$1)\; y_{ij} = \theta_{i1} + \theta_{i2}\left(t_j - 8\right) + e_{ij},\; e_{ij} \sim_{\text{iid}} \mathcal{N}\left(0, \sigma_e^2\right)$$

$$2)\; \boldsymbol{\theta}_i = \begin{pmatrix} \theta_{i1} \\ \theta_{i2} \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \alpha_0 + \alpha x_i \\ \beta_0 + \beta x_i \end{pmatrix}, \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix}\right]$$

$$x_i = I_{[i = "Fille"]}$$

Cas particulier: "Modèle intercept aléatoire"

$$\theta_{i2} = \beta_0 + \beta x_i;\; \theta_{i1} \sim \mathcal{N}\left(\alpha_0 + \alpha x_i, \sigma_a^2\right)$$

modèle "mixte classique"

$$y_{ij} = \left(\alpha_0 + \alpha x_i + a_i\right) + \left(\beta_0 + \beta x_i\right)\left(t_j - 8\right) + e_{ij}$$

$$y_{ij} = \underbrace{\alpha_0 + \beta_0\left(t_j - 8\right)}_{\text{profil moyen "garçon)}} + \underbrace{\left(\alpha + \beta\left(t_j - 8\right)\right)}_{\text{écart "fille-garçon"}} x_i + \underbrace{a_i}_{\text{intercept individuel aléatoire}} + e_{ij}$$

Winbugs/Modèle Hiérarchique

```
model {
for (i in 1:K) {
for (j in 1:n) {
    Y[i, j] ~ dnorm(eta[i, j], tau0)
    eta[i, j]<-intercept[i]+pente[i]*(age[j]-8)
    }
#intercept aléatoire
 intercept[i]~ dnorm(alfa[i], tau1)
 alfa[i]<-alpha0+alpha*sex[i]
 pente[i]<-beta0+beta*sex[i]
  }
```

Winbugs/Modèle Hiérarchique

$$\text{Quid de } \tau_0 = 1/\sigma_0^2, \tau_1 = 1/\sigma_1^2 \text{ ?}$$

2. Gamma/rappel

Gamma standard $X \sim \mathcal{G}(\alpha,1) \Leftrightarrow f(x) = I_{[x>0]} \dfrac{x^{\alpha-1}\exp(-x)}{\Gamma(\alpha)}$

où $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}\exp(-t)dt$: fonction gamma

$E(X) = Var(X) = \alpha$

Posons $y = x/\beta \Rightarrow dx/dy = \beta$

Gamma $Y \sim \mathcal{G}(\alpha,\beta) \Leftrightarrow g(y) = I_{[y>0]} \dfrac{\beta^{\alpha} y^{\alpha-1}\exp(-\beta y)}{\Gamma(\alpha)}$

$E(Y) = \alpha/\beta;\ Var(Y) = \alpha/\beta^2$

Khi-deux : $\frac{1}{2}\chi^2_\nu \sim \mathcal{G}(\nu/2,1) \Rightarrow \boxed{\chi^2_\nu \sim \mathcal{G}(\nu/2,1/2)}$

Gamma/densités

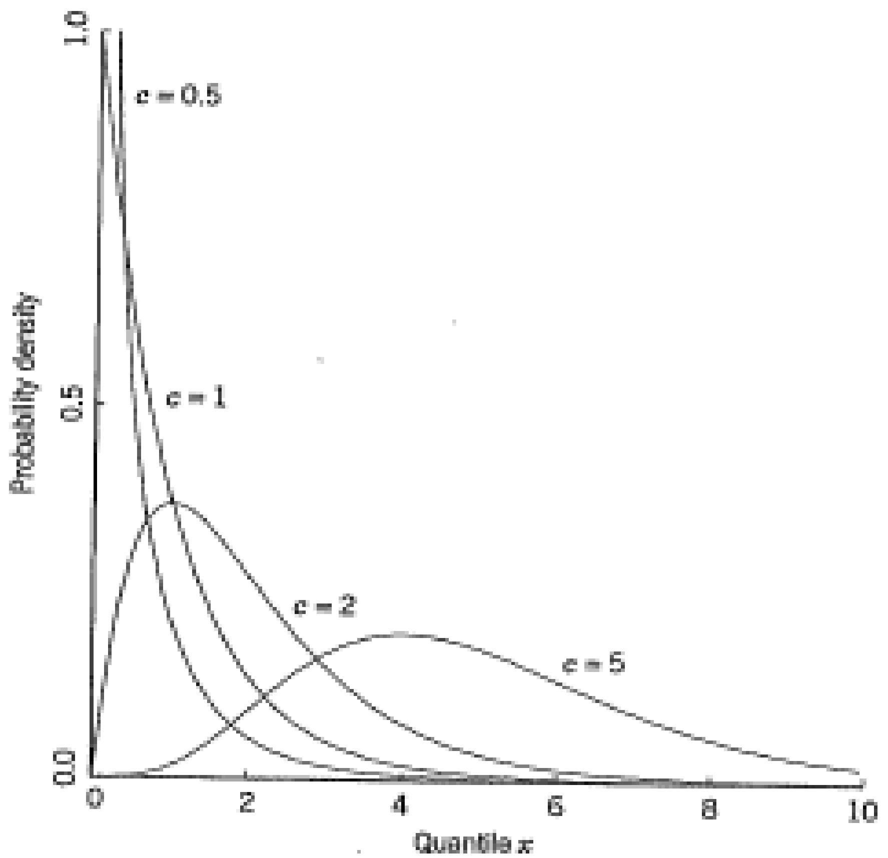


FIGURE 18.1. Probability density function for the gamma variate γ: 1, c.

A priori Gamma-inverse?

Argument de la stat fiduciaire via un pivot

Rappel: si $x_i \sim_{iid} N\left(\mu,\sigma^2\right); i=1,\ldots,N$

$$\underbrace{\left(N-1\right)}_{\eta} s^2 \sim \sigma^2 \chi^2_{(N\text{-}1)} \quad \text{où} \quad s^2 = \sum_{i=1}^{N}\left(x_i-\bar{x}\right)^2 / \left(N-1\right)$$

$$\frac{\eta s^2}{\sigma^2} \sim \chi^2_\eta \Leftrightarrow \boxed{\frac{1}{\sigma^2} \sim \frac{1}{\eta s^2}\chi^2_\eta}$$

Or $\chi^2_\eta \sim 2\mathcal{G}\left(½\eta,1\right) \sim \mathcal{G}\left(½\eta,½\right), \dfrac{1}{\sigma^2} \sim \dfrac{1}{\eta s^2}\mathcal{G}\left(½\eta,½\right)$

$\boxed{\sigma^{-2} \sim \mathcal{G}\left(½\eta,½\eta s^2\right)}$ avec $\mathrm{E}(\sigma^{-2}) = s^{-2}$

Remplacer s^2 par une valeur centrale donnée $\underline{\sigma}^2$

A priori Gamma-inverse?

$$Rappel : \sigma^{-2} \sim \mathcal{G}\left(½\eta, ½\eta\underline{\sigma}^2\right)$$

$$\Rightarrow E\left(\sigma^{-2}\right) = \underline{\sigma}^{-2}; Var(\sigma^{-2}) = 2\underline{\sigma}^{-4}/\eta$$

$$E(\sigma^2) = \frac{\eta}{\eta - 2}\underline{\sigma}^2; Var(\sigma^2) = \frac{2\eta^2\underline{\sigma}^4}{(\eta-2)^2(\eta-4)}$$

Options

1) $\sigma^2 \sim \mathcal{U}(0,\mathrm{A})$

2) $\sigma \sim ½A * \mathcal{C}(0,1) \Rightarrow \sigma \sim \mathcal{U}(0,.)$ si $\mathrm{A} \to +\infty$ Gelman, 2006

3) $\sigma^{-2} \sim \mathcal{G}\left(½\eta, ½\eta\underline{\sigma}^2\right)$

  $a)\eta$ et $\underline{\sigma}^2$ connus en particulier η petit ie 2

  $b)\eta \sim \mathrm{U}(0,\mathrm{H})$ et $\underline{\sigma}^2$ connus

4) $\sigma^{-2} \sim \mathcal{G}(\varepsilon,\varepsilon) \Leftrightarrow \left| \begin{array}{l} \ln\sigma^2 \sim \mathcal{U}[-\infty,+\infty] \text{ si } \varepsilon \to 0 \\ \pi\left(\sigma^2\right) \propto 1/\sigma^2 (Jeffreys) \end{array} \right.$

ε petit à calibrer en fonction de $\sum e^2 \left(\sum u^2\right)$

Choix de l'a priori: Gelman (2006)

Modèle paramétrique "étendu" (PX-EM, Liu et al, 1999)

$$y_{ij} = \mu + u_i + e_{ij}, u_i \sim_{iid} N\left(0, \sigma_u^2\right), e_{ij} \sim_{iid} N\left(0, \sigma_e^2\right)$$

$$u_i = \alpha u_i^* \Rightarrow \sigma_u = |\alpha| \sigma_u^*$$

a priori independants sur α *et* σ_u^{*2} (et conjugés)

$$\left.\begin{array}{r} |\alpha| \sim ½\mathcal{N}\left(0, A^2\right) \\ 1/\sigma_u^{2*} \sim \chi_\nu^2 = \mathcal{G}\left(\nu/2, 1/2\right) \end{array}\right| \Rightarrow \sigma_u \sim ½ \frac{\mathcal{N}\left(0, A^2\right)}{\sqrt{\chi_\nu^2}} = ½\mathrm{T}_\nu\left(0, A^2\right)$$

ν=1: $½A * \mathcal{C}(0,1)$: Cauchy; $A \to \infty \Rightarrow$ uniforme sur σ_u

Code Winbugs/Cas diffus 3a $\sigma^{-2} \sim \mathcal{G}\left(\frac{1}{2}\eta, \frac{1}{2}\eta\underline{\sigma}^2\right)$

```
# A priori
alpha0~ dnorm(0.0, 1.0E-6)
 alpha~ dnorm(0.0, 1.0E-6)

 beta0~ dnorm(0.0, 1.0E-6)
 beta~ dnorm(0.0, 1.0E-6)

tau0~ dgamma(f,g)
g<-f*var0.e
var0<-1/tau0
sig0<-pow(var0,0.5)

tau1~ dgamma(d,e)
e<-d*var1.e
var1<-1/tau1
sig1<-pow(var1,0.5)
```

Code Winbugs/Cas diffus 3a/Data&Inits

Data

```
list(K=25,n=4,var1.e=485,var0.e=100,d=2,f=2,
Y = structure(.Data = c(
210,200,215,230,
210,215,240,255,
205,NA,245,260,
235,245,250,265,
215,230,225,235,
200,NA,210,225,
215,225,230,250,
230,230,235,240,
200,NA,220,215,
165,NA,190,195,
245,250,280,280,

260,250,290,310,
215,NA,230,265,
230,225,240,275,
255,275,265,270,
200,NA,225,260,
245,255,270,285,
220,220,245,265,
240,215,245,255,
# 230,205,310,260,
275,280,310,315,
230,230,235,250,
215,NA,240,280,
# 170,NA,260,295,
225,255,255,260,
230,245,260,300,
220,NA,235,250),
.Dim = c(25, 4)),
age=c(8, 10,12,14),
sex=c(1,1,1,1,1,1,1,1,1,1,1,
0,0,0,0,0,0,0,0,0,0,0,0,0,0))
```

Inits

```
list(alpha0 = 200, alpha=0, beta0=10, beta=0 )
```

Résultats d'ensemble

Caractéristiques des lois a posteriori des composantes de variance
(données Pothoff et Roy ; 25 indiv ;Little & Rubin moins individus 20 et 24)

	σ_0^2	σ_1^2
1) Uniforme variance	109.4 (107.0) [101.34] (76.9 ;155.8)	480.3 (447.2) [406.85] (243.2 ;919.4)
2)Gelman-Cauchy	106,2 (104.0) [99.31] (75.0,149.6)	447.9 (418.3) [360.53] (229.1,818,8)
3) Diffus	106.0 (103.8) [98.34] (75.5 ;149.2)	436.7 (411.6) [375.01] (235.9 ;784.7)
4) G(ε,ε)	106.2 (103.9) [98.17] (75.2 ;150.3)	431.4 (403.7) [353.49] (224.3 ;800.7)
5) ML	99.68±*17.23*	361.45±*110.28*
6) REML	102.74±*18.03*	394.50±*124.96*

Espérance (Médiane) [Mode]
(a ;b)=Intervalle de crédibilité à 95%

1)U(0,5000) ; 2) Cauchy sur σ_1^2 et Jeffreys sur σ_0^2 ,3) η=2 σ_0^2 =100 ; σ_1^2 =485 ; 4) ε=1^E-6

5) et 6) Estimation±SE

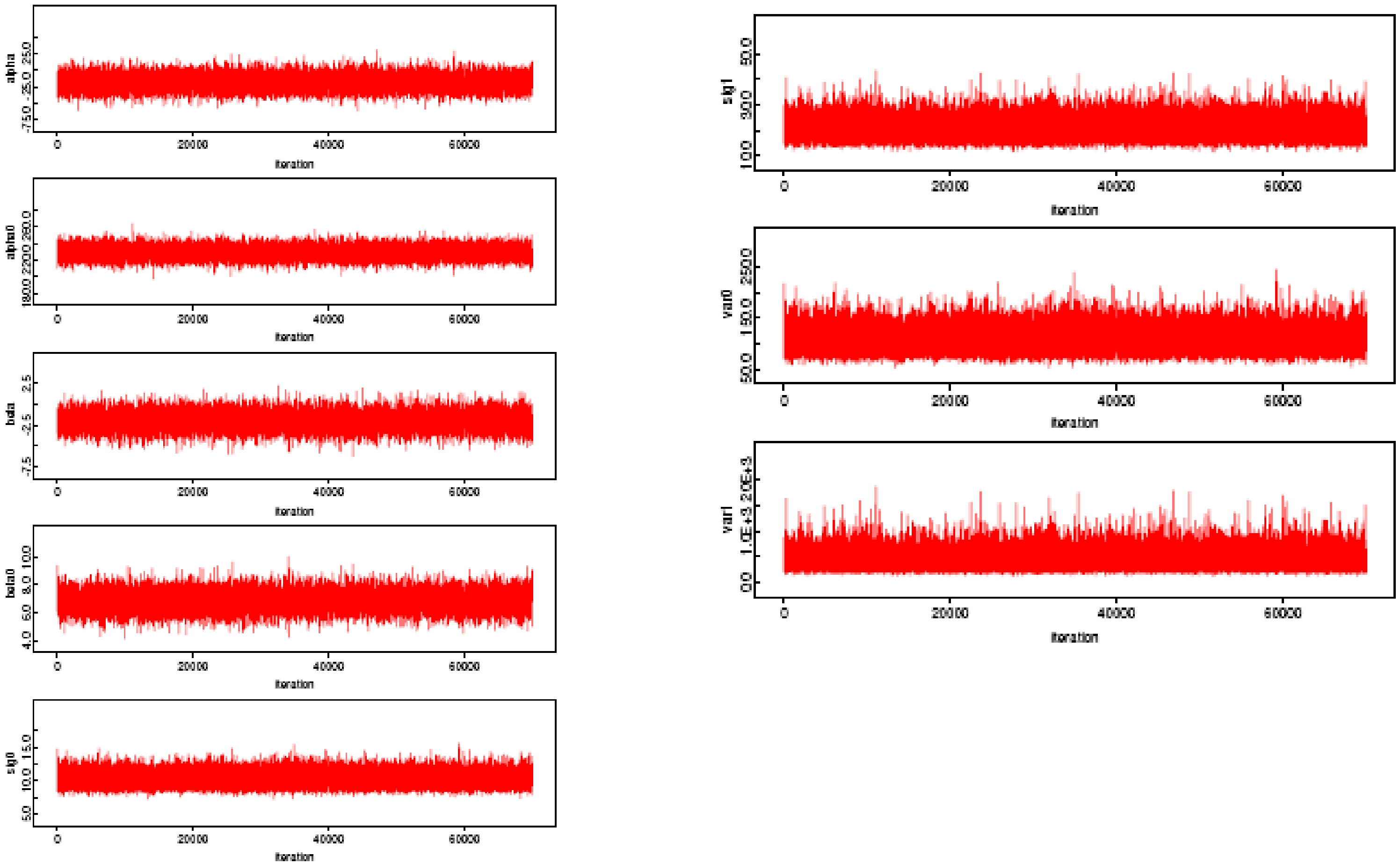
Sorties Openbugs/Option 1

Exemple Pothoff & Roy (25 individus) Priors sur les variances G(e,e) avec e=1.0E-6

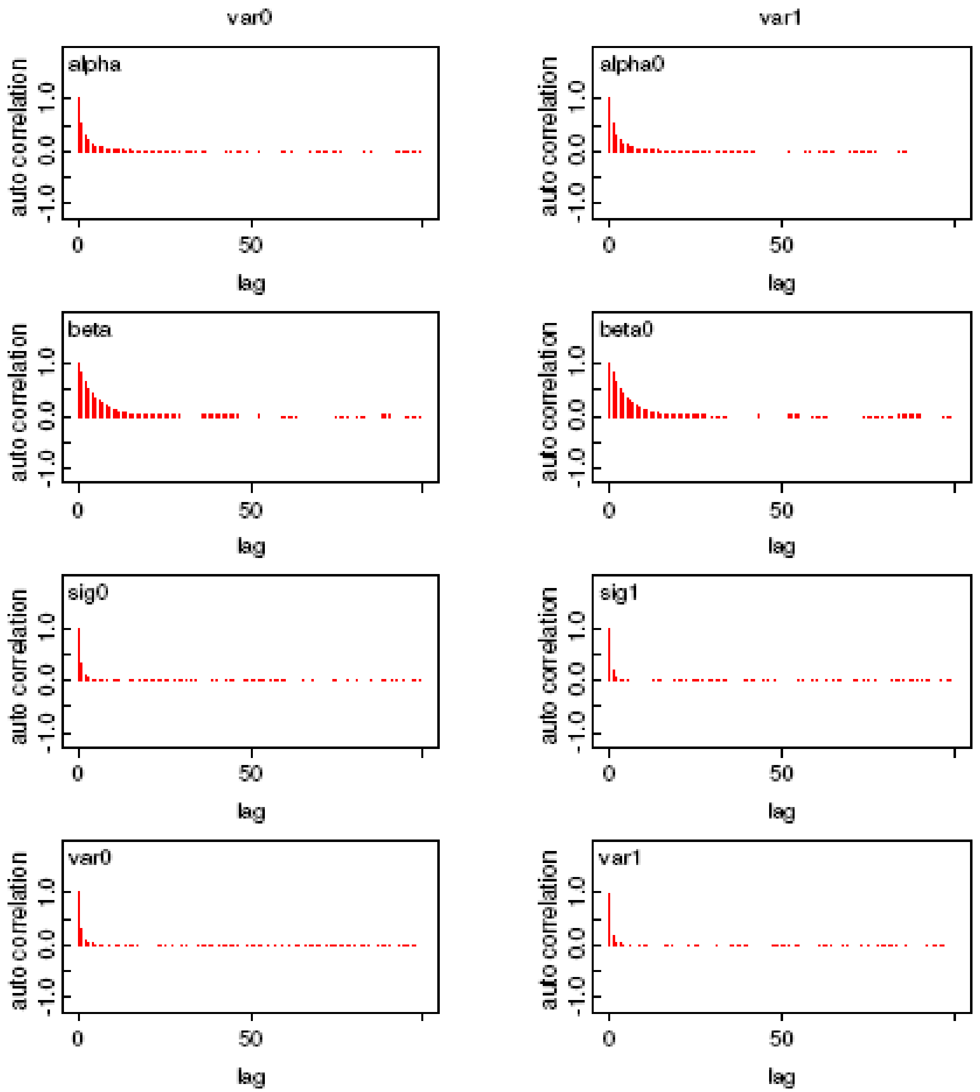
	mean	sd	MC_error	val2.5pc	median	val97.5pc	start	sample
alpha	-18.37	9.194	0.07296	-36.48	-18.39	-0.256	1	70000
alpha0	229.5	6.07	0.04719	217.6	229.5	241.4	1	70000
beta	-1.956	0.9466	0.01136	-3.798	-1.961	-0.07704	1	70000
beta0	6.869	0.6289	0.006976	5.625	6.868	8.113	1	70000
sig0	10.27	0.9165	0.00464	8.671	10.19	12.26	1	70000
sig1	20.49	3.404	0.01648	14.98	20.09	28.3	1	70000
var0	106.2	19.25	0.09759	75.18	103.9	150.3	1	70000
var1	431.4	149.7	0.7237	224.3	403.8	800.7	1	70000

	Dbar	Dhat	DIC	pD
Y	688.9	662.0	715.8	26.93
total	688.9	662.0	715.8	26.93

Sorties Openbugs/Option 1/Histoire

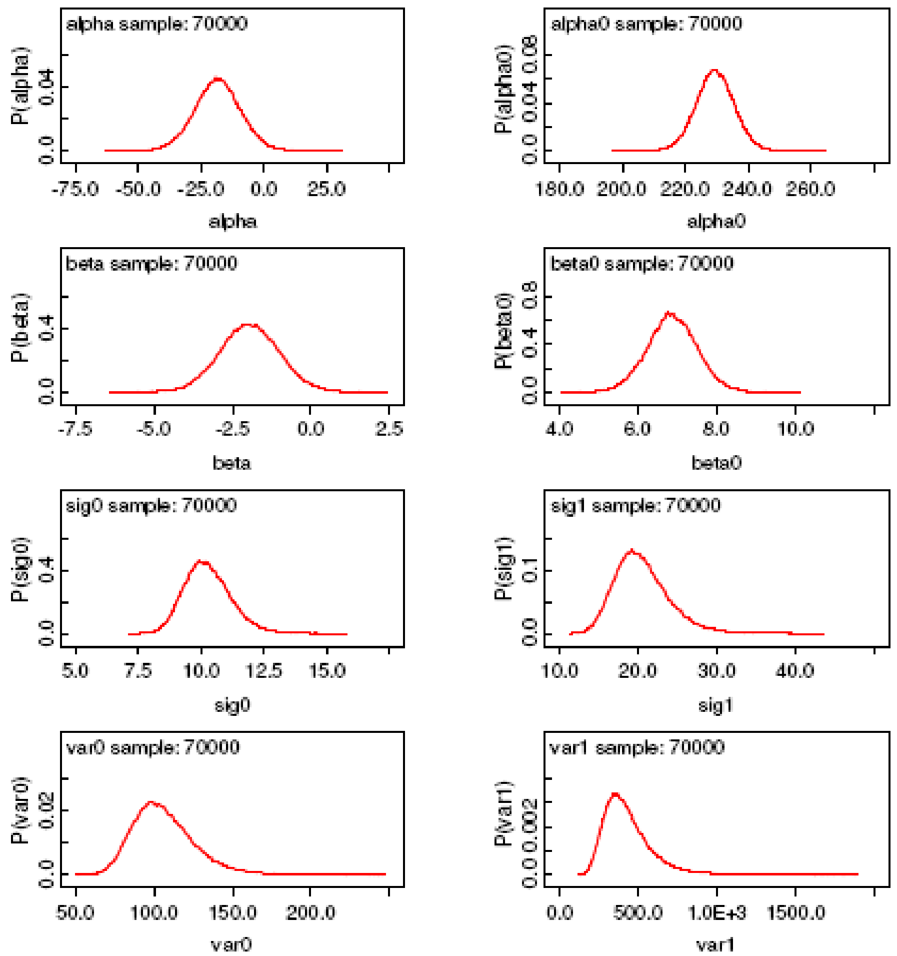


Sorties Openbugs/Option 1 /Autocorrélation

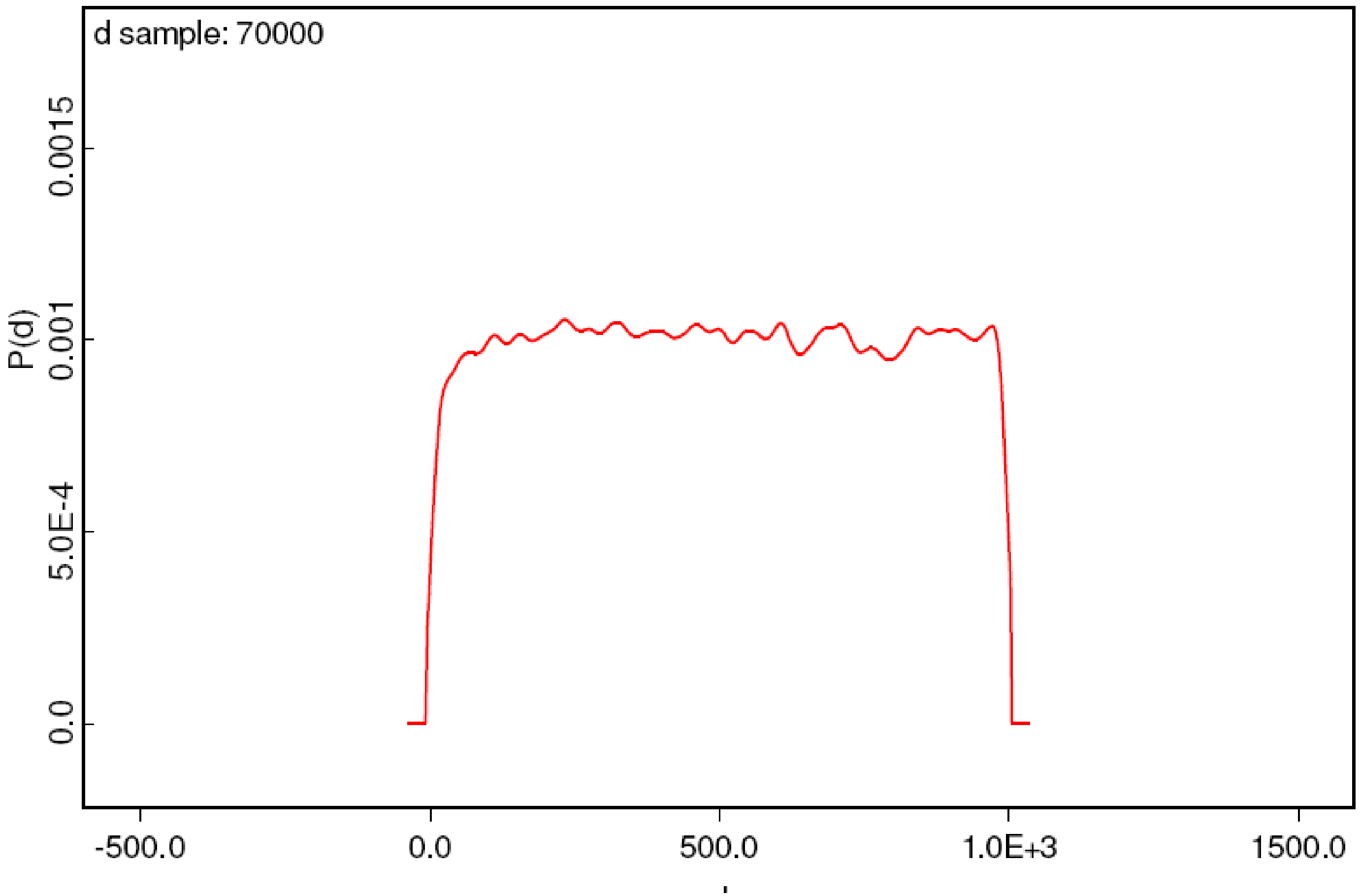


Sorties Openbugs/Option 1/Posterior

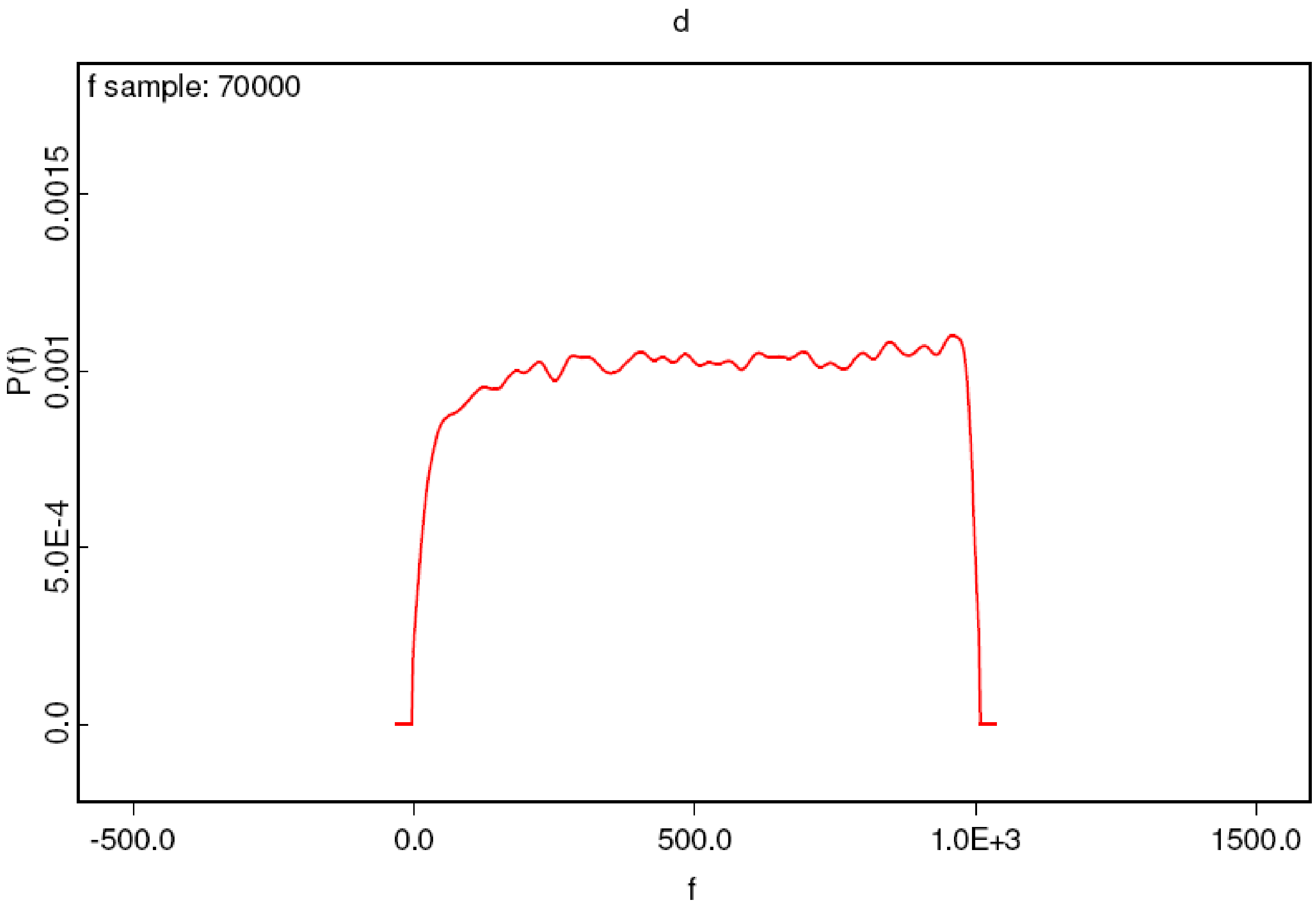
A posteriori des paramètres de dispersion: lois a priori dur les variances gamma(epsilon,epsilon)
Ex Pothoff & Roy: 25 individus + données manquantes de Little & Rubin.



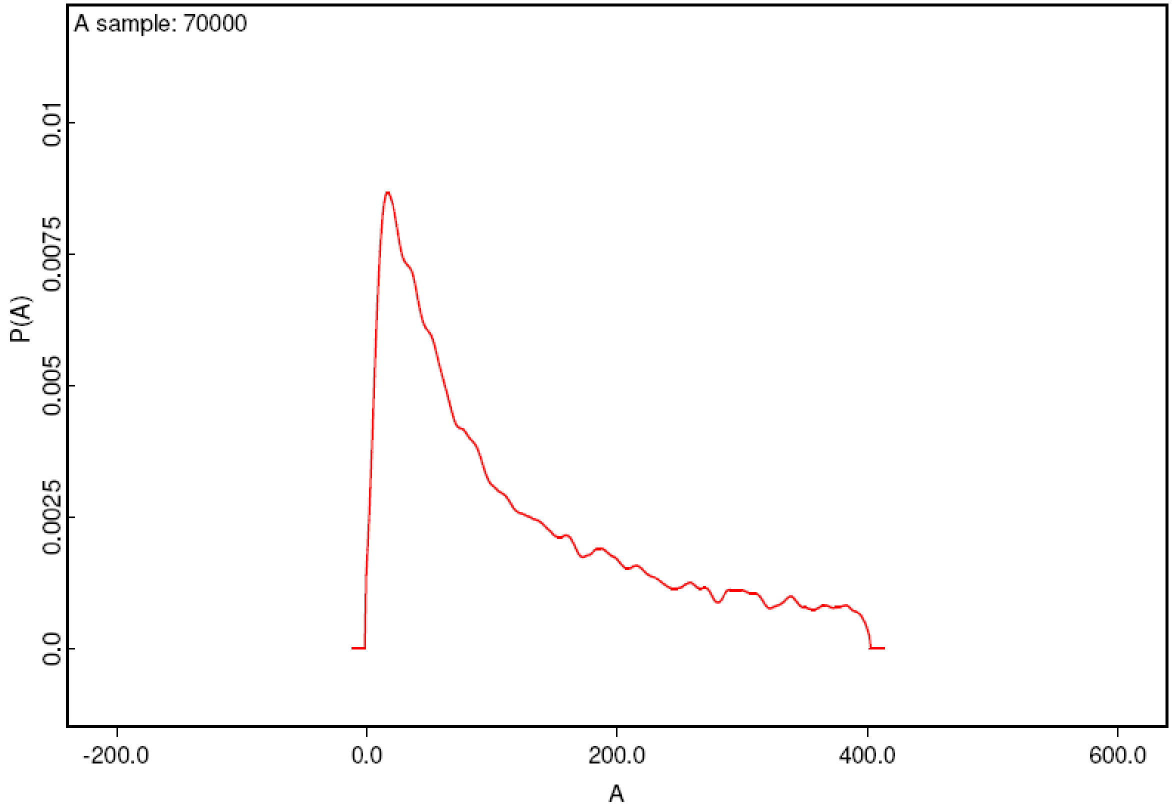
Sorties Openbugs/Option 3b eta/var résiduelle



Sorties Openbugs/Option 3b eta/var individuelle



Sorties Openbugs/Option 2/par A de Gelman



	mean	sd	MC_error	val2.5pc	median	val97.5pc	start	sample
A	117.9	102.5	3.194	7.318	82.26	368.4	5001	70000

Estimations ML & REML/SAS Proc Mixed

Analyse des données Pothoff & Roy par un modèle « intercept aléatoire » via SAS Proc Mixed 9.1 (25 individus et données manquantes de Little & Rubin)

Covariance Parameter Estimates ML

Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
Intercept	individu	361.45	110.28	3.28	0.0005
Residual		99.6819	17.2307	5.79	<.0001

Covariance Parameter Estimates REML

Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
Intercept	individu	394.50	124.96	3.16	0.0008
Residual		102.74	18.0300	5.70	<.0001

Conclusion

- Plusieurs choix possibles des a priori de variance avec un choix naturel dans la famille conjuguée gamma-inverse.
- Dans le cas du modèle mixte, l'inférence bayésienne sur les effets « fixes » et les prédictions prennent en compte l'incertitude sur les variances contrairement aux approches classiques basées sur ML et REML.

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